PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON THURSDAY, APRIL 25, 2024 – ON CARDINALITY AND

THE CONTINUUM Hypothesis

CLASS #28

Consider the History of Euclidean Geometry

Euclid ~300 BC publishes "The Elements"

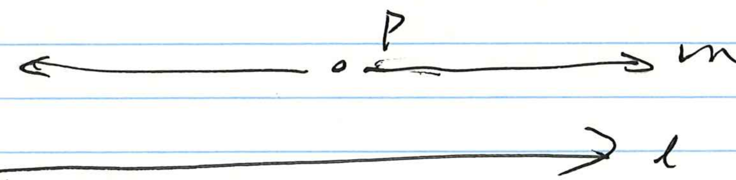
in Geometry is developed from

5 Postulates (axioms = Statements that are assumed to be true without proof)

The 5^{th} Postulate is the following:

Given a line ~~line~~ l and a point P not on line l,

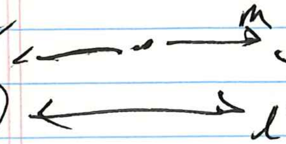
There exists one unique line m such that m passes through point P and m is parallel to l.



In the 1800's it was proved the 5^{th} Postulate is undecidable from the first ~~5~~ 4 Postulates. –
The 5^{th} Postulate is Independent of the first 4 postulates.

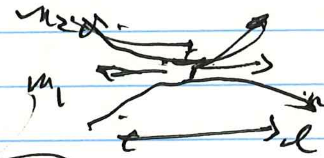
5th Postulate

Assume the 5th is True



Euclidean Geometry

Assume the 5th Postulate is False



Non-Euclidean Geometries

Def'n: Given a set A,

the Cardinality of A, $CARD(A)$,

is the collection of all sets X such that

X and A can be put into one-to-one correspondence, i.e.

There exists a one-to-one correspondence f

$f: A \to X$ and $f^{-1}: X \to A$.

Ex: If $A = \{a, b, c\}$,

$CARD(A)$ is collection of all ~~set~~ sets with three elements.

$CARD(A) = 3$

$CARD(\mathbb{Z}^+)$ = the collection of all countably infinite sets.

$CARD(\mathbb{Z}^+) = \aleph_0$ (Aleph Nulle)

$CARD(\mathbb{R}) = \mathbf{c}$ (The Cardinality of the Continuum)

Def'n For Two Sets A and B,

we say $\mathrm{CARD}(A) < \mathrm{CARD}(B)$

$\Longleftrightarrow$

① There exists a one-to-one function from A to B

and

② there does not exist an onto function from A to B.

Theorem (4U.13) is (That (0,1) is uncountable)

essentially says

$\mathrm{CARD}(\mathbb{Z}^+) < \mathrm{CARD}(\mathbb{R})$

$\aleph_0 < c$

Theorem: For any Set S,

$\mathrm{CARD}(S) < \mathrm{CARD}(\mathcal{P}(S))$

Result:

$$\mathrm{CARD}(\mathbb{Z}^+) < \mathrm{CARD}(\mathbb{R}) < \mathrm{CARD}(\mathcal{P}(\mathbb{R}))$$
$$< \mathrm{CARD}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$$
$$< \mathrm{CARD}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R}))))$$
$$< \vdots$$

The Continuum Hyposthesis (proposed in Early 1900's)

There does not exist a set X such that

$$\mathbb{Z}^+ \subsetneq X \subsetneq \mathbb{R} \quad \text{such that}$$

a proper subset of ...

$$\aleph_0 = \mathrm{CARD}(\mathbb{Z}^+) < \mathrm{CARD}(X) < \mathrm{CARD}(\mathbb{R}) = \mathfrak{c}$$

In 1963, Paul Cohen proved that The Continuum Hypothesis is Undecidable – It is independant of the Axioms of Mathematics

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Continuum Hypothesis:

Assume it true

$\left\{ \text{Most of Today's mathematics} \right\}$

Assume it False

$\left\{ ? \right\}$